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2002 J. Phys. A: Math. Gen. 35 L543

(http://iopscience.iop.org/0305-4470/35/38/101)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) L543–L549 PII: S0305-4470(02)38681-5

LETTER TO THE EDITOR

Logarithmic corrections to finite-size spectrum of $SU(N)$ symmetric quantum chains

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Received 25 June 2002
Published 12 September 2002
Online at stacks.iop.org/JPhysA/35/L543

Abstract
We consider $SU(N)$ symmetric one-dimensional quantum chains at finite temperature. For such systems the correlation lengths, ground state energy and excited state energies are investigated in the framework of conformal field theory. The possibility of different types of excited states is discussed. Logarithmic corrections to the ground state energy and different types of excited states in the presence of a marginal operator are calculated. The known results for $SU(2)$ and $SU(4)$ symmetric systems follow from our general formula.

PACS numbers: 75.10.Jm, 75.40.Gb

One of the fundamental models of solid state physics is the Heisenberg model of insulating magnets. In the one-dimensional case (‘spin chains’), the spin-$\frac{1}{2}$ Heisenberg models have been studied extensively: most of our understanding of their quantum critical behaviour is based on the Bethe ansatz solution for the ground state and excitation spectrum [1, 2], mapping to the sine-Gordon theory [3], non-Abelian bosonization [4] and mapping to the sigma model [5]. Although spin-$\frac{1}{2}$ Heisenberg chains are $SU(2)$ symmetric systems, fruitful generalizations have been accomplished in two different directions: (a) enlarging the representation of the $SU(2)$ group to study quantum chains with higher spins and (b) introducing higher symmetry groups such as $SU(N)$.

Here we consider generalizations of type (b) and investigate how higher symmetry affects the ground state properties (equations (8)–(10)) and finite-size spectrum of quantum ‘spin’ chains. Earlier studies of Affleck [6] show that any one-dimensional system with $SU(N)$ symmetry is critical, and at very low energy scale these models are equivalent to $(N-1)$ free massless bosons. These free bosons, when viewed in the framework of two-dimensional conformal field theory, are the primary fields of the $SU(N)_{k=1}$ WZNW model. Adopting this model, we give an explicit derivation of logarithmic corrections to the finite-size spectrum of $SU(N)$ symmetric quantum chains. Logarithmic shifts in excited states

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energy levels have been theoretically observed for $SU(2)$ and $SU(4)$ symmetric systems away from the $T = 0$ quantum critical point [7–12]. The known results for $N = 2$ and $N = 4$ follow from our general formulae (obtained in equations (27)–(28) and the paragraphs below equation (30)).

A one-dimensional $SU(N)$ symmetric quantum chain of length L is described by the Hamiltonian [13]

$$H = \sum_{j=1}^{n} \sum_{A=1}^{N^2-1} S_j^A S_{j+1}^A. \tag{1}$$

Here $n = L/a_0$ is the total number of discrete points (a_0 being the lattice spacing) and S_j^A are the $(N^2 - 1)$ generators of the $SU(N)$ Lie algebra at each lattice site j. For convenience, the interaction strength and the lattice spacing a_0 have been set equal to 1 in equation (1). At each site j, the generators S_j^A can be represented by N 'flavours' of fermions, ψ_{aj} $(a = 1, \ldots, N)$,

$$S_j^A = \sum_{a,b=1}^{N} \psi_j^{a\dagger} (T^A)_a^b \psi_{bj} - I/N \tag{2}$$

where I is the identity operator, and T^A are a complete set of $(N^2 - 1)$ traceless normalized matrices so that $\mathrm{Tr}[T^A T^B] = \frac{1}{2}\delta^{AB}$. Equation (2) satisfies the constraint that at each site the total number of fermions is conserved, i.e. $\sum_{a=1}^{N} \psi_j^{a\dagger} \psi_{ja} = 1$.

For such a fermionic system, the theory can be bosonized using non-Abelian bosonization at low temperatures. In the continuum limit, the bosonized Hamiltonian (H_{eff}) [4] can then be written in terms of the Kac–Moody currents,

$$H_{\mathrm{eff}} \approx v_s \sum_{A=1}^{N^2-1} \int \mathrm{d}x \left[J_L^A J_L^A + J_R^A J_R^A + 2 J_L^A J_R^A \right] \tag{3}$$

where the normal ordered Kac–Moody currents for the left (with Fermi momentum $k_F < 0$) and the right moving (with $k_F > 0$) fermions are defined as

$$J_L^A = : \psi_L^{a\dagger} (T^A)_a^b \psi_{Lb} : \qquad J_R^A = : \psi_R^{a\dagger} (T^A)_a^b \psi_{Rb} : . \tag{4}$$

At zero temperature ($T = 0$) the interaction term, $\sum_{A=1}^{N^2-1} J_L^A J_R^A$, in equation (3) renormalizes to zero, and the sum of the first two terms in the Hamiltonian that are quadratic in left and right moving currents corresponds to the $SU(N)_{k=1}$ WZNW model. The fundamental unitary $N \times N$ matrix field g of the WZNW model is given by

$$g_b^a = (\mathrm{const}) : \psi_L^{a\dagger} \psi_{Rb} : . \tag{5}$$

The field g transforms into the fundamental representation of $SU(N)_L \times SU(N)_R$ which describes the exact symmetry of the Hamiltonian in equation (3) at zero temperature. It is known that this fermionic theory is equivalent to a theory of $(N - 1)$ free massless bosons at the criticality with velocities v_s; they correspond to $(N - 1)$ excitation modes of the $SU(N)$ symmetric quantum chain that oscillates at different values of k_F [6]. Furthermore, these oscillating modes are primary fields of the $SU(N)_{k=1}$ WZNW model and their scaling dimensions (Δ_p) can be obtained from [14]

$$\Delta_p = \frac{2C_p}{C_{\mathrm{adj}} + 1}. \tag{6}$$

For $SU(N)$, C_p $(p = 1, 2, \ldots, N - 1)$ is the eigenvalue of the Casimir operator in the pth fundamental representation (having a Young tableau with p boxes in a single column), and

$C_{\rm adj} = N$ is the eigenvalue of the Casimir operator in the adjoint representation (having a two-column Young tableau with $(N-1)$ boxes in the first column and one box in the second column). In a highest weight (Λ) representation of $SU(N)$, the corresponding Casimir eigenvalue is given by [15]

$$C_\Lambda = \frac{(\theta,\theta)}{2}\left[m\left(N-\frac{m}{N}\right)+\sum_{i=1}^{r_0}(b_i)^2-\sum_{i=1}^{c_0}(a_i)^2\right] \tag{7}$$

where θ is the highest weight corresponding to the adjoint representation and *is normalized to* 1, m is the total number of boxes in the Young tableau with r_0 rows of length $b_1, b_2, \ldots, b_{r_0}$ and c_0 columns of length $a_1, a_2, \ldots, a_{c_0}$. Using this formula we find $C_p = p(N-p)(N+1)/2N$ and $C_{\rm adj} = N$. Hence, the scaling dimensions (Δ_p) of primary fields of the $SU(N)_{k=1}$ WZNW model are given by

$$\Delta_p = \frac{2C_p}{C_{\rm adj}+1} = \left[\frac{p(N-p)}{N}\right]. \tag{8}$$

For example, in the case of $SU(4)$ the three oscillating components have scaling dimensions $\left(\frac{3}{4}, 1, \frac{3}{4}\right)$ (for $p = 1, 2, 3$). For $SU(N)$, the mode (dominant) that oscillates at $k_F = 2\pi/N$ has a scaling dimension $(1-1/N)$ ($p = 1$ or $(N-1)$ in this case).

Finite-size corrections to the Heisenberg chain with $SU(2)$ and $SU(4)$ symmetry have been studied using conformal field theory [7, 17]. The relevance of studying finite-size chains is twofold. One can not only compare the theoretical results with numerical simulations and experiments which are limited to the finite size of the system but can also study the finite-temperature behaviour of the system by identifying the finite size in the imaginary time direction, which corresponds to finite temperature. To obtain the finite-size corrections of a one-dimensional chain of length L and with periodic boundary conditions, we first introduce a conformal mapping from the infinite plane (with coordinate z) to the cylinder (with coordinate w) via $w = (L/2\pi)\ln z$. Identifying the length as the inverse of the temperature $(L = v_s/T)$ the finite-temperature results [17] of the ground state energy E_0 can be generalized to the $SU(N)$ symmetric system,

$$E_0(T) = E_0(0) - \frac{\pi T(N-1)}{6v_s}. \tag{9}$$

Here $E_0(0)$ refers to the ground state energy at zero temperature. The thermodynamic quantities such as specific heat and entropy can now be obtained by taking the appropriate derivatives with respect to the temperature.

Other quantities of interest are the finite-temperature corrections to the correlation lengths (ξ) of different modes. These inverses of the correlation lengths, ξ^{-1}, are signatures of energy gaps $(E_n - E_0)$ between the ground state and the lowest lying excited states (E_n) that are created by the finite temperature of the system. Using the general formula for the scaling dimension (equation (8)), we obtain ξ_p^{-1} of the pth staggered mode,

$$\begin{aligned}\xi_p^{-1} &\equiv E_n^p - E_0 \\ &= \left(\frac{2\pi T}{v_s}\right)\Delta_p = \left(\frac{2\pi T}{v_s}\right)\left[\frac{p(N-p)}{N}\right].\end{aligned} \tag{10}$$

The temperature dependence of the correlation lengths is in fact modified by logarithmic corrections in the presence of marginal operators in the theory [17]. The generic form of the Hamiltonian at the critical point containing a marginal operator $\phi(x,t)$ is

$$H = H^* + g_0\int \mathrm{d}x\,\phi \tag{11}$$

where g_0 is the coupling constant and H^* is the Hamiltonian at the fixed point. In our case (equation (3)), the normalized marginally irrelevant operator is

$$\phi = -D \sum_{A=1}^{N^2-1} J_L^A J_R^A. \tag{12}$$

For such a marginally irrelevant operator, the Hamiltonian at the critical point ($T = 0$) becomes equal to the fixed-point Hamiltonian. In equation (12), D is the normalization constant to be determined from the two-point correlator of ϕ. The operator product expansion (OPE) of the $J_{L(R)}^A$ with any normalized Kac–Moody primary field χ is given by [16]

$$J_L^A(z)\chi(z') = \frac{J_{0,L}^A}{2\pi\mathrm{i}(z-z')}\chi(z') + \cdots \tag{13}$$

$$J_R^A(\bar{z})\chi(\bar{z}') = \frac{J_{0,R}^A}{2\pi\mathrm{i}(\bar{z}-\bar{z}')}\chi(\bar{z}') + \cdots \tag{14}$$

where the operators $J_{0,L}^A$ and $J_{0,R}^A$ are the generators of the global $SU(N)_L \times SU(N)_R$ transformations, and satisfy the characteristic equations $J_{0,L}^A|\chi(z')\rangle = -T_L^A|\chi(z')\rangle$ and $J_{0,R}^A|\chi(\bar{z}')\rangle = |\chi(\bar{z}')\rangle T_R^A$. Note that $\sum_A \left(J_{0,L}^A\right)^2$ and $\sum_A \left(J_{0,R}^A\right)^2$ are the Casimir operators of $SU(N)_L$ and $SU(N)_R$ groups, respectively. The two-point correlators of the left currents (for $k = 1$) are

$$\left\langle J_L^A(z) J_L^B(z')\right\rangle = -\frac{\mathrm{Tr}\left[T_L^A T_L^B\right]}{4\pi^2(z-z')^2} = -\frac{\delta^{AB}}{8\pi^2(z-z')^2} \tag{15}$$

$$\left\langle J_R^A(\bar{z}) J_R^B(\bar{z}')\right\rangle = \frac{\delta^{AB}}{8\pi^2(\bar{z}-\bar{z}')^2}. \tag{16}$$

Using these results we explicitly calculate

$$\langle\phi(z,\bar{z})\phi(z',\bar{z}')\rangle = \left(\frac{D}{8\pi^2}\right)^2 \frac{(N^2-1)}{(z-z')^2(\bar{z}-\bar{z}')^2} \tag{17}$$

and then compare it to the standard conformal field theory result, i.e. $\langle\phi(z,\bar{z})\phi(z',\bar{z}')\rangle = |z-z'|^{-2}|\bar{z}-\bar{z}'|^{-2}$, to obtain the value of the constant,

$$D = \frac{8\pi^2}{\sqrt{N^2-1}}. \tag{18}$$

Thus the normalized irrelevant marginal operator is given by

$$\phi(z,\bar{z}) = -\frac{8\pi^2}{\sqrt{N^2-1}} \sum_{A=1}^{N^2-1} J_L^A(z) J_R^A(\bar{z}). \tag{19}$$

Perturbation to the normalized excited state (ϕ_n) energies due to the marginal operator can now be calculated [17] from

$$\delta(E_n - E_0) = g_0 \int \mathrm{d}x\, \langle\phi_n|\phi|\phi_n\rangle \tag{20}$$

where ϕ and ϕ_n are Virasoro primary fields generated by applying Fourier modes of J_L^A and J_R^A on Kac–Moody primary fields. For large length (equivalently, small temperature), we may replace the coupling g_0 by its renormalization group improved value (up to the log–log term) [18],

$$g_0(T) = \left(\frac{1}{\pi b \ln(T_0/T)}\right)\left[1 - \frac{1}{2\ln(T_0/T)}\ln[\ln(T_0/T)]\right]. \tag{21}$$

Here T_0 is the model-dependent parameter of the system and the coefficient b is defined via the following three-point correlator:

$$\langle \phi(z_1, \bar{z}_1)\phi(z_2, \bar{z}_2)\phi(z_3, \bar{z}_3)\rangle = -b/|z_{12}|^2|z_{23}|^2|z_{13}|^2. \tag{22}$$

Substituting this in equation (20) we obtain

$$\delta(E_n - E_0) = \left(\frac{2\pi T}{v_s \ln(T_0/T)}\right)\left(\frac{2b_n}{b}\right)\left[1 - \frac{1}{2\ln(T_0/T)}\ln[\ln(T_0/T)]\right]. \tag{23}$$

The coefficient b_n is again defined through the three-point correlator,

$$\langle \phi_n(z_1, \bar{z}_1)\phi(z_2, \bar{z}_2)\phi_n(z_3, \bar{z}_3)\rangle = -b_n/|z_{12}|^2|z_{23}|^2|z_{13}|^{2x_n-2}. \tag{24}$$

Here x_n is the scaling dimension of the Virasoro primary field ϕ_n. Substituting equation (19) in equation (24) and using the OPEs as in equations (13) and (14) it follows that b_n is directly proportional to the sum of the product of the eigenvalues of the generators $J^A_{0,L}$ and $J^A_{0,R}$:

$$b_n = -\frac{2}{\sqrt{N^2-1}}\sum_{A=1}^{N^2-1} T^A_L T^A_R. \tag{25}$$

To evaluate $\sum_A T^A_L T^A_R$, we observe that the full symmetry, $SU(N)_L \times SU(N)_R$, of the quantum chain at the critical point is broken by the presence of the marginal operator $\phi(z, \bar{z})$. Only the diagonal $SU(N) \subset SU(N)_L \times SU(N)_R$ is an exact symmetry of the quantum chain. Under this subgroup, the representation $V_L \otimes V_R$ of $SU(N)_L \times SU(N)_R$ decomposes into the direct sum of various irreducible subrepresentations. If an excited state $(|\phi_n\rangle)$ belongs to a highest weight subrepresentation $V \subset V_L \otimes V_R$ and C is the corresponding Casimir invariant of the diagonal $SU(N)$ in V, then we have [19]

$$\sum_{A=1}^{N^2-1} T^A_L T^A_R = \frac{1}{2}[C - C_L - C_R] \tag{26}$$

where C_L and C_R are the Casimir invariants of $SU(N)_L$ and $SU(N)_R$ in the highest weight representations V_L and V_R, respectively. Therefore, using equations (25) and (26) we find

$$b_n = -\frac{1}{\sqrt{N^2-1}}[C - C_L - C_R]. \tag{27}$$

The above formula may also be used to evaluate the renormalization group coefficient b (equation (22)): since $\phi(z, \bar{z})$ is a Virasoro primary field of conformal dimensions (1,1), we set $\phi_n(z, \bar{z}) = \phi(z, \bar{z})$ and $x_n = 2$ in equation (24), and hence $b_n = b$. This can be seen as follows. The Virasoro primary fields J^A_L and J^A_R of conformal dimensions $(1, 0)$ and $(0, 1)$ transform as the adjoint representations V^{adj}_L and V^{adj}_R of $SU(N)$, and since V^{adj}_R is conjugate to V^{adj}_L the direct sum decomposition of $V^{\mathrm{adj}}_L \otimes V^{\mathrm{adj}}_R$ under the diagonal $SU(N) \subset SU(N)_L \times SU(N)_R$ must contain a unique singlet. Hence, the Virasoro primary field $\phi(z, \bar{z})$ in equation (19) transforms as this singlet representation and we have $C = 0$, $C_L = C_R = N$ in equation (27). This implies

$$b = \frac{2N}{\sqrt{N^2-1}}. \tag{28}$$

For example, in the case of Heisenberg spin-chain with $SU(2)$ symmetry, $b = 4/\sqrt{3}$ and for the spin-orbital model with $SU(4)$, $b = 8/\sqrt{15}$. Our result for $SU(4)$ is new. Together with equation (21), the constant b also determines the correction of $O(g_0^3)$ in the ground state energy,

$$E_0(T) - E_0(0) = -\left(\frac{\pi T}{6v_s}\right)\left[(N-1) + 2\pi^3 b g_0^3\right]. \tag{29}$$

To determine the logarithmic shifts in the excited states energy levels, we need the ratio $2b_n/b$ in equation (23). From equations (27) and (28) we get

$$\left(\frac{2b_n}{b}\right) = -\frac{1}{N}[C - C_L - C_R]. \tag{30}$$

To evaluate b_n (and hence $2b_n/b$) we must know the excited states. In $SU(N)$ invariant quantum chains, the low lying excited states ($|\phi_n\rangle$) correspond to $(N-1)$ primary fields of the $SU(N)_{k=1}$ WZNW model with the scaling dimensions Δ_p. These fields transform as $(q, \bar{q})$ representations of $SU(N)_L \times SU(N)_R$, where

$$q = \frac{N(N-1)\cdots(N-p+1)}{p!}$$

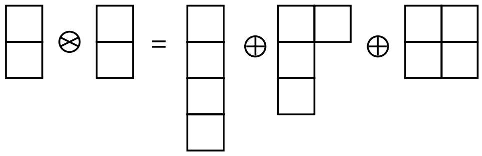
is the dimension of the pth fundamental representation of $SU(N)$ for $p = 1, 2, \ldots, N-1$. For instance, the lowest excited states correspond to the fundamental primary field g with the scaling dimensions $\Delta_1 = (1 - 1/N)$. This field transforms under the $(N, \bar{N})$ representation which decomposes under the diagonal $SU(N)$ into the adjoint and singlet representations. For the excited states, $\mathrm{Tr}\,[g\,T^A]$, belonging to the adjoint representation, we have $C = N$, $C_L = C_R = (N^2-1)/2N$ which implies $b_n = -1/(N\sqrt{N^2-1})$ and $2b_n/b = -1/N^2$. For example, in the case of $SU(2)$ the ratio $2b_n/b = -\frac{1}{4}$ [7], and for $SU(3)$ and $SU(4)$ this is $-\frac{1}{9}$ and $-\frac{1}{16}$ respectively.

For the excited state, $\mathrm{Tr}\,g$, belonging to the singlet representation, we have $C = 0$, $C_L = C_R = (N^2-1)/2N$. In this case, $b_n = \sqrt{N^2-1}/N$ and the $2b_n/b = 1 - 1/N^2$. This result is new. In case of $SU(2)$, the value $2b_n/b = \frac{3}{4}$ has been previously obtained [7] but for $SU(3)$, $2b_n/b = \frac{8}{9}$ and for $SU(4)$, $2b_n/b = \frac{15}{16}$ are the predictions from our general formula.

We consider one more application of formula (30) of current interest—the $SU(4)$ symmetric quantum chain described by the $SU(4)_{k=1}$ WZNW model. In this case, to compute logarithmic corrections to the excited states energy we note that there are three primary fields with scaling dimensions $\Delta_p = \frac{3}{4}, 1, \frac{3}{4}$ for $p = 1, 2, 3$ respectively, as seen from equation (8). The case of $p = 1$ (and $p = 3$), as discussed above, is the fundamental field g (and its Hermitian conjugate $\bar{g}$) which transforms under the $(4, \bar{4})$ (and $(\bar{4}, 4)$) representation of $SU(4)_L \times SU(4)_R$. From equation (10), the next lowest energy excited states correspond to the primary field operator (denoted by Ψ) with $\Delta_2 = 1$. The field Ψ transforms under the $(6, 6)$ representation of $SU(4)_L \times SU(4)_R$. The $(6, 6)$ representation decomposes as the direct sum of a singlet, an adjoint and a 20-dimensional representation (as in figure 1) under the diagonal $SU(4)$.

We now compute the ratio $2b_n/b$ for the excited states corresponding to the 20-dimensional representation which has a Young tableau with two rows and two columns. For this representation, the Casimir invariant C in equation (30) is obtained from formula (7): we find $C = 6$, and $C_L = C_R = C_{p=2} = \frac{5}{2}$. Thus, for the excited states corresponding to Ψ in the 20-dimensional subrepresentation of $(6, 6)$, we have $2b_n/b = -\frac{1}{4}$.

In summary, we have studied the finite-size spectrum for one-dimensional $SU(N)$ symmetric quantum chains using both conformal field theory and representation theory of $SU(N)$. We have calculated in general the scaling dimensions of all the oscillating modes,



(6) (6) (1) (15) (20)

Figure 1. Young tableau for the decomposition of the (6, 6) representation of $SU(4)$. The number in the parentheses denotes the dimension of the corresponding representation.

and obtained the ground state energy as well as correlation lengths of the staggered modes for a finite-size system with $SU(N)$ symmetry. The possibilities of different types of excited states are also briefly discussed and a general formula to compute the logarithmic correction to the excited state energies has been derived. The existing results for $N = 2, 4$ agree with the predictions from our general formula.

The authors acknowledge discussions with F C Zhang and M Ma. KM was partially supported by PRF no 33611-AC6 and acknowledges the partial support from Berea College. MM is grateful to Professor P D Scholten of Miami University for his crucial support and encouragement.

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